Two divisors of $(n^2+1)/2$ summing up to $\delta n+\varepsilon$, for δ and ε even

ELementare und Analytische Zahlentheorie, 2014
Hildesheim, Germany

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Recent Results

- In [1], Ayad and Luca have proved that there does not exist an odd integer $n > 1$ and two positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = n + 1$.
- In [2], Dujella and Luca have dealt with a more general issue, where $n + 1$ was replaced with an arbitrary linear polynomial $\delta n + \varepsilon$, where $\delta > 0$ and ε are given integers.
- Since $d_1 + d_2 = \delta n + \varepsilon$, then there are two cases: it is either $\delta \equiv \varepsilon \equiv 1 \pmod 2$, or $\delta \equiv \varepsilon + 2 \equiv 0 \text{ or } 2 \pmod 4$.
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Introduction

We deal with the second case, the case where

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- We completely solve cases when $\delta = 2, \ \delta = 4$ and $\varepsilon = 0$.
  - We prove that there exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n + \varepsilon$ for $\varepsilon \equiv 0 \pmod 4$.
  - We prove an analoguos result for $\varepsilon \equiv 2 \pmod 4$ and divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 4n + \varepsilon$.
  - We also prove that there exist infinitely many odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n$.
  - In case when $\delta \geq 6$ is a positive integer of the form $\delta = 4k + 2, \ k \in \mathbb{N}$ we prove that there does not exist an odd integer n such that there exists a pair of divisors d_1, d_2 of $\frac{n^2+1}{2}$ with the property $d_1 + d_2 = \delta n$.

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The case $\delta = 2$, $\varepsilon \equiv 0 \pmod 4$

Theorem

If $\varepsilon \equiv 0 \pmod 4$, then there exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n + \varepsilon$.

The case $\delta = 2$, $\varepsilon \equiv 0 \pmod 4$

- We want to find a positive odd integer n and positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n + \varepsilon$.
- Let $g = \gcd(d_1, d_2)$. We can write $d_1 = gd_1', d_2 = gd_2'$. Since $gd_1'd_2' = \operatorname{lcm}(d_1, d_2)$ divides $\frac{n^2+1}{2}$, we conclude that there exists a positive integer d such that

$$d_1 d_2 = \frac{g(n^2+1)}{2d}.$$

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From the identity

$$(d_2-d_1)^2=(d_1+d_2)^2-4d_1d_2,$$

we can easily obtain

$$(d_2-d_1)^2=(2n+\varepsilon)^2-4\frac{g(n^2+1)}{2d},$$

$$d(4d-2g)(d_2-d_1)^2=(4d-2g)^2n^2+4(4d-2g)d\varepsilon n+4d^2\varepsilon^2-8dg-2\varepsilon^2dg+4g^2. \tag{1}$$

For $X=(4d-2g)n+2d\varepsilon$, $Y=d_2-d_1$, the equation (1) becomes

$$X^2-d(4d-2g)Y^2=8dg+2\varepsilon^2dg-4g^2.$$

The case $\delta = 2$, $\varepsilon \equiv 0 \pmod 4$

For $g = 1$ the previous equation becomes

$$X^2 - 2d(2d-1)Y^2 = 2d(4+\varepsilon^2) - 4. \tag{2}$$

The equation (2) is a Pellian equation. The right-hand side of (2) is nonzero.

Our goal is to make the right-hand side of (2) a perfect square. That condition can be satisfied by taking $d = \frac{1}{8}\varepsilon^2 - \frac{1}{2}\varepsilon + 1$.

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Pellian equation (2) becomes

$$X^2 - 2d(2d-1)Y^2 = \left(\frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)\right)^2. \qquad (3)$$

Now, like in [2], we are trying to solve (3).

Let

$$X = \frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)U, \quad Y = \frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)V.$$

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The equation (3) becomes

$$U^2 - 2d(2d-1)V^2 = 1. \tag{4}$$

Equation (4) is a Pell equation which has infinitely many positive integer solutions (U, V), and consequently, there exist infinitely many positive integer solutions (X, Y) of (3).
The least positive integer solution of (4) can be found using the continued fraction expansion of number

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We can easily get

$$\sqrt{2d(2d-1)} = [2d-1; \overline{2, 4d-2}].$$

All positive solutions of (4) are given by (U_m, V_m) for some $m \geq 0$.

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Generally, solutions of (4) are generated by recursive expressions

$$U_0 = 1,\ \ U_1 = 4d-1,\ \ U_{m+2} = 2(4d-1)U_{m+1} - U_m,$$

$$V_0 = 0,\ \ V_1 = 2,\ \ V_{m+2} = 2(4d-1)V_{m+1} - V_m,\ \ m \in \mathbb{N}_0. \quad (5)$$

By induction on m, one gets that $U_m \equiv 1 \pmod{(4d-2)}$, $m \geq 0$.

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It remains to compute the corresponding values of n which arise from

$$X = (4d-2)n + 2d\varepsilon, \;\; X = \frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)U.$$

We obtain

$$n = \frac{\frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)U - 2d\varepsilon}{4d-2}.$$

We want the above number n to be a positive integer.

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Congruences

$$\frac{1}{2}(\varepsilon^2-2\varepsilon+4)U-2d\varepsilon \equiv 4d+\varepsilon-2-2d\varepsilon \equiv -(2d-1)\varepsilon \equiv 0 \pmod{(4d-2)},$$

show us that all numbers n generated in the specified way are integers.

The case $\delta = 2$, $\varepsilon \equiv 0 \pmod 4$

The first few values of number n, which we get from U_1, U_2, U_3, are

$$\begin{cases} n = \frac{1}{2}(\varepsilon^2 - 3\varepsilon + 6), \\ d_1 = 1, \\ d_2 = \varepsilon^2 - 2\varepsilon + 5. \end{cases}$$

$$\begin{cases} n = \frac{1}{2}(\varepsilon^4 - 6\varepsilon^3 + 20\varepsilon^2 - 33\varepsilon + 34), \\ d_1 = \varepsilon^2 - 2\varepsilon + 5, \\ d_2 = \varepsilon^4 - 6\varepsilon^3 + 19\varepsilon^2 - 30\varepsilon + 29. \end{cases}$$

$$\begin{cases} n = \frac{1}{2}(\varepsilon^6 - 10\varepsilon^5 + 50\varepsilon^4 - 148\varepsilon^3 + 281\varepsilon^2 - 323\varepsilon + 198), \\ d_1 = \varepsilon^4 - 6\varepsilon^3 + 19\varepsilon^2 - 30\varepsilon + 29, \\ d_2 = \varepsilon^6 - 10\varepsilon^5 + 49\varepsilon^4 - 142\varepsilon^3 + 262\varepsilon^2 - 292\varepsilon + 169. \end{cases}$$

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Proof of this theorem will be slightly different from the previous proof.

Instead of assuming that $\varepsilon \equiv 2 \pmod 4$, we will distiguish two cases: in one case we will be dealing with $\varepsilon \equiv 6 \pmod 8$ and we will apply strategies from [2] and in the other case we will be dealing with $\varepsilon \equiv 2 \pmod 8$ and we will use different methods in obtaining results.

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We start with the case when $\varepsilon \equiv 6 \pmod 8$.
Let $g = \gcd(d_1, d_2)$, $d_1 = gd_1'$, $d_2 = gd_2'$ and d is a positive integer which satisfies the equation

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From the identity

$$(d_2 - d_1)^2 = (d_1 + d_2)^2 - 4d_1 d_2,$$

we obtain

$$d(16d-2g)(d_2-d_1)^2 = (16d-2g)^2 n^2 + 8(16d-2g)d\varepsilon n + 16d^2\varepsilon^2 - 32dg - 2\varepsilon^2 dg + 4g^2. \tag{6}$$

Let $X = (16d-2g)n + 4d\varepsilon$, $Y = d_2 - d_1$. Equation (6) becomes

$$X^2 - 2d(8d-g)Y^2 = 32dg + 2\varepsilon^2 dg - 4g^2. \tag{7}$$

For $g = 1$ the previous expression becomes

$$X^2 - 2d(8d-1)Y^2 = 2d(16+\varepsilon^2) - 4. \tag{8}$$

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Our goal is to make the right-hand side of (8) a perfect square.
That condition can be satisfied by taking

$$d=\frac{1}{32}\varepsilon^2-\frac{1}{8}\varepsilon+\frac{5}{8}.$$

Pellian equation (8) becomes

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Let

$$X = \frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W, \quad Y = \frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)Z.$$

The equation (9) becomes

$$W^2 - 2d(8d-1)Z^2 = 1. \tag{10}$$

The equation (10) is a Pell equation which has infinitely many positive integer solutions (W, Z), and consequently, there exist infinitely many positive integer solutions (X, Y) of (9).
The least positive integer solution of (10) can be found using the continued fraction expansion of number $\sqrt{2d(8d-1)}$.
We can easily get

$$\sqrt{2d(8d-1)} = [4d-1; \overline{1, 2, 1, 8d-2}].$$

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All positive solutions of (10) are given by (W_m, Z_m) for some $m \geq 0$. Generally, solutions of (10) are generated by recursive expressions

$$W_0 = 1, \ \ W_1 = 16d - 1, \ \ W_{m+2} = 2(16d-1)W_{m+1} - W_m,$$

$$Z_0 = 0, \ \ Z_1 = 4, \ \ Z_{m+2} = 2(16d-1)Z_{m+1} - Z_m, \ \ m \in \mathbb{N}_0.$$

The case $\delta = 4$, $\varepsilon \equiv 2 \pmod 4$

By induction on m, one gets that

$$W_m \equiv 1 \pmod{(16d-2)}, \quad m \geq 0.$$

It remains to compute the corresponding values of n which arise from

$$X = (16d-2)n + 4d\varepsilon, \quad X = \frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W.$$

We obtain

$$n = \frac{\frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W - 4d\varepsilon}{16d-2}.$$

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The congruences

$$\frac{1}{4}(\varepsilon^2-2\varepsilon+16)W-4d\varepsilon \equiv 8d-1+\frac{\varepsilon}{2}-4d\varepsilon \equiv (8d-1)(1-\frac{\varepsilon}{2}) \equiv 0 \pmod{(16d-2)}$$

show us that all numbers n generated in the specified way are integers.

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The first few values of number n, which we get from W_1, W_2, W_3, are

$$\begin{cases} n = \frac{1}{4}(\varepsilon^2 - 3\varepsilon + 18), \\ d_1 = 1 \\ d_2 = \varepsilon^2 - 2\varepsilon + 17. \end{cases}$$

$$\begin{cases} n = \frac{1}{4}(\varepsilon^4 - 6\varepsilon^3 + 44\varepsilon^2 - 105\varepsilon + 322), \\ d_1 = \varepsilon^2 - 2\varepsilon + 17, \\ d_2 = \varepsilon^4 - 6\varepsilon^3 + 43\varepsilon^2 - 102\varepsilon + 305. \end{cases}$$

$$\begin{cases} n = \frac{1}{4}(\varepsilon^6 - 10\varepsilon^5 + 86\varepsilon^4 - 388\varepsilon^3 + 1529\varepsilon^2 - 3155\varepsilon + 5778), \\ d_1 = \varepsilon^4 - 6\varepsilon^3 + 43\varepsilon^2 - 102\varepsilon + 305, \\ d_2 = \varepsilon^6 - 10\varepsilon^5 + 85\varepsilon^4 - 382\varepsilon^3 + 1486\varepsilon^2 - 3052\varepsilon + 5473. \end{cases}$$

The case $\delta = 4$, $\varepsilon \equiv 2 \pmod 4$

Now, we deal with the case when $\varepsilon \equiv 2 \pmod 8$.
Let $\varepsilon = 8k+2,\ \ k \in \mathbb{N}_0$. For $g = \frac{1}{4}\varepsilon^2 + 4$ and $g = d_1$, the equation (7) becomes

$$X^2 - 2d(8d-g)Y^2 = \frac{2d-1}{4}\varepsilon^4 + 8\varepsilon^2(2d-1) + 64(2d-1).$$

The right-hand side of the equation will be a perfect square if $2d-1$ is a perfect square. Motivated by the experimental data, we take

$$d = \frac{1}{512}\varepsilon^4 - \frac{1}{64}\varepsilon^3 + \frac{7}{64}\varepsilon^2 - \frac{5}{16}\varepsilon + \frac{41}{32}.$$

The case $\delta = 4$, $\varepsilon \equiv 2 \pmod 4$

Now, we deal with the case when $\varepsilon \equiv 2 \pmod 8$.
Let $\varepsilon = 8k+2$, $k \in \mathbb{N}_0$. For $g = \frac{1}{4}\varepsilon^2 + 4$ and $g = d_1$, the equation (7) becomes

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The case $\delta = 4$, $\varepsilon \equiv 2 \pmod 4$

We get

$$2d - 1 = 16k^4 + 8k^2 + 1 = (4k^2+1)^2.$$

So, the equation (7) becomes

$$X^2 - 2d(8d-g)Y^2 = \left(\frac{1}{32}(\varepsilon^2+16)(\varepsilon^2-4\varepsilon+20)\right)^2. \qquad (11)$$

We consider the corresponding Pell equation

$$U^2 - 2d(8d-g)V^2 = 1. \qquad (12)$$

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The case $\delta = 4$, $\varepsilon \equiv 2 \pmod 4$

Let (U_0, V_0) be the least positive integer solution of (12). That equation has infinitely many solutions. From (12) we get that

$$U^2 \equiv 1 \pmod{(16d - 2g)}.$$

We deal with the case where $g = d_1 = \frac{1}{4}\varepsilon^2 + 4$ and from the experimental data we can set

$$d_2 = d_1^2 - 16kd_1, \quad k \in \mathbb{N}_0.$$

For $Y = d_2 - d_1$ we get

$$Y = \left(\frac{1}{4}\varepsilon^2 + 4\right)^2 - (2\varepsilon - 3)\left(\frac{1}{4}\varepsilon^2 + 4\right) = \frac{\varepsilon^4}{16} - \frac{\varepsilon^3}{2} + \frac{11\varepsilon^2}{4} - 8\varepsilon + 28.$$

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The case $\delta = 4$, $\varepsilon \equiv 2 \pmod 4$

From (11), we obtain:

$$X = \frac{(\varepsilon^2+16)(\varepsilon^6-16\varepsilon^5+140\varepsilon^4-768\varepsilon^3+3120\varepsilon^2-8704\varepsilon+14400)}{2048}.$$

We claim that X satisfies the congruence

$$X \equiv 4d\varepsilon \pmod{(16d-2g)}. \tag{13}$$

Indeed,

$$16d-2g = \frac{\varepsilon^4}{32}-\frac{\varepsilon^3}{4}+\frac{5\varepsilon^2}{4}-5\varepsilon+\frac{25}{2},$$

$$X-4d\varepsilon = \left(\frac{\varepsilon^4}{32}-\frac{\varepsilon^3}{4}+\frac{5\varepsilon^2}{4}-5\varepsilon+\frac{25}{2}\right)\left(\frac{\varepsilon^4}{64}-\frac{\varepsilon^3}{8}+\frac{13\varepsilon^2}{16}-\frac{9\varepsilon}{4}+9\right).$$

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The case $\delta = 4$, $\varepsilon \equiv 2 \pmod 4$

From $n = \frac{X-4d\varepsilon}{16d-2g}$, we get

$$n = \frac{\varepsilon^4}{64} - \frac{\varepsilon^3}{8} + \frac{13\varepsilon^2}{16} - \frac{9\varepsilon}{4} + 9 = 64k^4 + 28k^2 + 7,$$

and we see that n is an odd integer.
Thus, if we define

$$(X_0, Y_0) = \left(\frac{(\varepsilon^2+16)(\varepsilon^6-16\varepsilon^5+140\varepsilon^4-768\varepsilon^3+3120\varepsilon^2-8704\varepsilon+14400)}{2048},\right.$$

$$\left.\frac{1}{16}(\varepsilon^2+16)(\varepsilon^2-8\varepsilon+28)\right),$$

we see that (X_0, Y_0) is a solution of (11) which satisfies the congruence (13).

The case $\delta = 4$, $\varepsilon \equiv 2 \pmod 4$

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The case $\delta = 4$, $\varepsilon \equiv 2 \pmod 4$

- We have proved that for every $\varepsilon \equiv 2 \pmod 8$ there exists at least one odd integer n which satisfies the conditions of this Theorem.
- Our goal is to prove that there exist infinitely many such integers n that satisfy the properties of this Theorem.

The case $\delta = 4$, $\varepsilon \equiv 2 \pmod 4$

If (X_0, Y_0) is a solution of (11), solutions of (11) are also

$$(X_i, Y_i) = \left(X_0 + \sqrt{2d(8d-g)}Y_0\right)\left(U_0 + \sqrt{2d(8d-g)}V_0\right)^{2i}, \quad i = 0, 1, 2, \ldots \tag{14}$$

From the equation (14), we get

$$X_i \equiv U_0^{2i} X_0 \equiv X_0 \equiv 4d\varepsilon \pmod{16d - 2g}.$$

So, there are infinitely many solutions (X_i, Y_i) of (11) that satisfy the congruence (13).

The case $\delta = 4$, $\varepsilon \equiv 2 \pmod 4$

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The case $\delta = 4$, $\varepsilon \equiv 2 \pmod 4$

Therefore, by

$$n = \frac{X_i - 4d\varepsilon}{16d - 2g},$$

we get infinitely many integers n with the required properties. It is easy to see that number n defined in this way is odd. Indeed, we have $16d - 2g \equiv 2 \pmod 4$, $X_0 \equiv 2 \pmod 4$, and since (12) implies that U_0 is odd and V_0 is even, we get from (13) that

$$X_i - 4d\varepsilon \equiv X_i \equiv U_0^{2i} X_0 \equiv X_0 \equiv 2 \pmod 4,$$

so n is odd.

The case $\varepsilon = 0$, $\delta = 2$

Proposition

There exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n$. These solutions satisfy $\gcd(d_1, d_2) = 1$ and $d_1 d_2 = \frac{n^2+1}{2}$.

The case $\varepsilon = 0$, $\delta = 2$

- We want to find a positive odd integer n and positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n$.
- Let $g = \gcd(d_1, d_2)$. Then $g|(2n)$ and $g|(n^2+1)$ which implies that $g|((2n)^2+4)$ so we can conclude that $g|4$.
- Because g is the greatest common divisor of d_1, d_2 and d_1, d_2 are odd numbers, we can also conclude that g is an odd number.
- So, $g = 1$.
- Like we did in the proofs of the previous theorems, we define a positive integer d which satisfies the equation $d_1 d_2 = \frac{n^2+1}{2d}$.

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- Like we did in the proofs of the previous theorems, we define a positive integer d which satisfies the equation $d_1 d_2 = \frac{n^2+1}{2d}$.

The case $\varepsilon = 0$, $\delta = 2$

From the identity

$$(d_2-d_1)^2=(d_1+d_2)^2-4d_1d_2,$$

we can easily obtain

$$d(d_2-d_1)^2=4n^2d-2n^2-2.$$

Let $d_2-d_1=2y$, so we get

$$(2d-1)n^2-2dy^2=1. \tag{15}$$

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The case $\varepsilon = 0$, $\delta = 2$

We will use the next lemma, which is Criterion 1 from [3] to check if there exists a solution for (15).

Lemma

Let $a > 1$, b be positive integers such that $\gcd(a, b) = 1$ and $D = ab$ is not a perfect square. Moreover, let (u_0, v_0) denote the least positive integer solution of the Pell equation

$$u^2 - Dv^2 = 1.$$

Then equation $ax^2 - by^2 = 1$ has a solution in positive integers x, y if and only if

$$2a|(u_0 + 1) \text{ and } 2b|(u_0 - 1).$$

The case $\varepsilon = 0$, $\delta = 2$

We want to solve the Pell equation

$$U^2 - 2d(2d-1)V^2 = 1, \tag{16}$$

where $n = U,\ \ y = V$.
The continued fraction expansion of the number $\sqrt{2d(2d-1)}$ is already known from Theorem 1 where we have obtained

$$\sqrt{2d(2d-1)} = [2d-1; \overline{2, 4d-2}].$$

The least positive integer solution of the Pell equation (16) is $(4d-1, 2)$. In our case, we want to find solutions of (15), so we apply Lemma which gives us conditions that have to be fulfilled.

The case $\varepsilon = 0$, $\delta = 2$

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The case $\varepsilon=0$, $\delta=2$

It has to be that

$$2(2d-1)|4d \text{ and } 4d|(4d-2),$$

which is not true for $d\in\mathbb{N}$. So, for Pellian equation (15) there are no integer solutions (n,y) when $a=2d-1>1$.
Finally, we have to check the remaining case for $a=1$, which is the case that is not included in Lemma.

If $a=2d-1=1$, then $d=1$.

The case $\varepsilon = 0$, $\delta = 2$

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The case $\varepsilon = 0$, $\delta = 2$

From (15) and $d = 1$, we get the Pell equation

$$n^2 - 2y^2 = 1, \tag{17}$$

which has infinitely many solutions $n = U_m,\ \ y = V_m,\ \ m \in \mathbb{N}_0$ where

$$U_0 = 1,\ \ U_1 = 3,\ \ U_{m+2} = 6U_{m+1} - U_m,$$

$$V_0 = 0,\ \ V_1 = 2,\ \ V_{m+2} = 6V_{m+1} - V_m,\ \ m \in \mathbb{N}_0.$$

The case $\varepsilon = 0$, $\delta = 2$

The first few values (U_i, V_i) are

$(U_0, V_0) = (1, 0)$, $(U_1, V_1) = (3, 2)$, $(U_2, V_2) = (17, 12)$, $(U_3, V_3) = (99, 70), \dots$.

From those solutions we can easily generate (n, d_1, d_2)

$$(n, d_1, d_2) = (3, 1, 5),\ \ (17, 5, 29),\ \ (99, 29, 169), \dots.$$

We have proved that in this case is $g = 1$ and $d = 1$, so we conclude that numbers d_1 and d_2 are coprime and that $d_1 d_2 = \frac{n^2+1}{2}$.

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The case $\varepsilon = 0$, $\delta \equiv 2 \pmod 4, \quad \delta \geq 6$

Theorem

Let $\delta \geq 6$ be a positive integer such that $\delta = 4k+2, k \in \mathbb{N}$. Then there does not exist a positive odd integer n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n$.

The case $\varepsilon = 0$, $\delta \equiv 2 \pmod 4,\ \ \delta \geq 6$

- Suppose on the contrary that this is not so and let the number δ be the smallest positive integer $\delta = 4k+2,\ \ k \in \mathbb{N}$ for which there exists an odd integer n and a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n$.
- Let $g = \gcd(d_1, d_2) > 1$. Since $d_1 = gd_1',\ \ d_2 = gd_2'$, it follows that $g|(n^2+1)$ and $g|(\delta n)$ and we conclude that $g|((\delta n)^2 + \delta^2)$, which implies that $g|\delta^2$.
- This means that g and δ have a common prime factor p.

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The case $\varepsilon = 0$, $\delta \equiv 2 \pmod 4$, $\delta \geq 6$

- Suppose on the contrary that this is not so and let the number δ be the smallest positive integer $\delta = 4k+2$, $k \in \mathbb{N}$ for which there exists an odd integer n and a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n$.
- Let $g = \gcd(d_1, d_2) > 1$. Since $d_1 = gd_1'$, $d_2 = gd_2'$, it follows that $g|(n^2+1)$ and $g|(\delta n)$ and we conclude that $g|((\delta n)^2 + \delta^2)$, which implies that $g|\delta^2$.
- This means that g and δ have a common prime factor p.

The case $\varepsilon = 0$, $\delta \equiv 2 \pmod 4,\ \delta \geq 6$

- Let $d_1 = pd_1'', d_2 = pd_2'', \delta = p\delta''$. Then, we have $pd_1'' + pd_2'' = p\delta'' n$, so we can conclude $d_1'' + d_2'' = \delta'' n$ where d_1'', d_2'' are divisors of $\frac{n^2+1}{2}$.
- It is clear that $\delta'' < \delta$ and if it also satisfies $\delta'' \neq 2$, the existence of the number δ'' contradicts the minimality of δ.
- So, if $\delta'' \neq 2$, then we must have $g = 1$.
- If $\delta'' = 2$, it follows from Proposition 1 that $\gcd(d_1'', d_2'') = 1$ and $d_1'' d_2'' = \frac{n^2+1}{2}$.
- But, $\gcd(d_1, d_2) = pd_1'' d_2''$ should be a divisor of $\frac{n^2+1}{2}$ which is not possible because $p > 1$.
- So, in this case we also conclude that $g = 1$.

The case $\varepsilon = 0$, $\delta \equiv 2 \pmod 4$, $\delta \geq 6$

- Let $d_1 = pd_1'', d_2 = pd_2'', \delta = p\delta''$. Then, we have $pd_1'' + pd_2'' = p\delta'' n$, so we can conclude $d_1'' + d_2'' = \delta'' n$ where d_1'', d_2'' are divisors of $\frac{n^2+1}{2}$.
- It is clear that $\delta'' < \delta$ and if it also satisfies $\delta'' \neq 2$, the existence of the number δ'' contradicts the minimality of δ.
- So, if $\delta'' \neq 2$, then we must have $g = 1$.
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From the identity

$$(d_2-d_1)^2=(d_1+d_2)^2-4d_1d_2,$$

and using $g=1$, we obtain

$$(\delta^2 d-2)n^2-d(d_2-d_1)^2=2.$$

We set $(d_2-d_1)=2y$ (number d_2-d_1 is an even number because d_1,d_2 are odd integers), and we get

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If we divide both sides by 2, we will get

$$(2d(2k+1)^2-1)n^2-2dy^2=1.$$

We define $\delta'=\frac{\delta}{2}=2k+1$, so we deal with

$$(2\delta'^2 d-1)n^2-2dy^2=1. \tag{18}$$

We will prove by applying Lemma that the above Pell equation (18) has no solutions.

To be able to apply Lemma, we have to deal with an equation of the form

$$x^2-Dy^2=1.$$

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We need to find the least positive integer solution of the equation

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For that purpose we find the continued fraction expansion of the number

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$$\sqrt{2d(2\delta'^2 d-1)}=[2d\delta'-1;\overline{1,2\delta'-2,1,2(2d\delta'-1)}].$$

So, the least positive integer solution is
$(p_3,q_3)=(u_0,v_0)=(4\delta'^2 d-1,2\delta')$ and we apply Lemma.

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We get

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We can easily see that $4d|(4\delta'^2 d - 2)$ if and only if $4d|2$ which is not possible because $d \in \mathbb{N}$.

So, the equation (18) has no solutions.

We have proved that there does not exist a positive odd integer n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n$.

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Acknowledgement

I would like to thank Professor Andrej Dujella for many valuable suggestions and a great help.

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